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Hawking Radiation of the Kerr–Newman Black Hole

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Abstract Considering the unfixed background space-time and self-gravitational interaction, we review the Hawking radiation of the Kerr–Newman black hole by Hamilton–Jacobi method. The result shows the tunneling probability is related to the change of Bekenstein–Hawking entropy and the radiation spectrum deviates from the precisely thermal one, which is in accordance with Parikh and Wilczek's result and gives another method to study the Hawking radiation of the black hole.

Keywords Self-gravitational interaction · Tunneling probability · Hamilton–Jacobi equation

Stephen Hawking's great academic discovery that the collapsing body radiates thermally [4, 5] not only resolved the antinomies existing in the black hole thermodynamics, but also opened out the intrinsic correlation among Quantum Theory, Thermodynamics and Gravitation Theory. When first proved the existence of the black hole radiation, he described it as a quantum tunneling process triggered by vacuum fluctuations near the horizon. According to this scenario, quite a few people studied the Hawking radiation of black holes in the past several decades. There are several methods to derive the Hawking radiation and most of them based on Quantum Field Theory on the fixed background space-time [3, 19, 20, 22, 25].

Recently, a semi-classical method to describe the Hawking radiation as tunneling process, where a particle moves in dynamical geometry, was proposed by Kraus, Parikh and Wilczek [9, 10, 16–18]. In their methodology, they pointed out the potential barrier is created by the outgoing particle itself; thereby the cause of the mechanism corresponding to the potential barrier was resolved. The energy conservation and self-gravitational interaction often neglected in former treatments were taken into account. In addition, a key point

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is the Painlevé coordinate was introduced to eliminate the coordinate singular. This coordinate is very convenient to study Hawking radiation of slowly evaporating black holes. In the process, the derivation of the radiation particle's action is crux. To get it, they should solve the Hamilton equations. Following this method, quite a few fruits have been achieved and all of these results supported Parikh and Wilczek's opinion [6, 12, 13, 23, 24, 26, 27]. In 2005, Zhang and Zhao et al. extended their work to the Hawking radiation of massive and charged particles and made a great deal of success [7, 26–28], which has effective significance on the further cognition and research on black holes. In the same year, a different method was introduced by Angheben et al. [2, 8]. It is called Hamilton–Jacobi method. The virtues of this method consist in the action is derived by Hamilton–Jacobi equation; therefore one can avoid introducing the Painlevé coordinate and exploring the motion equations of the radiation particles. But they ignored the unfixed background space-time and self-gravitational interaction so that the derived radiation spectrum only is leading term.

In this paper, considering the self-gravitational interaction as well as energy conservation, charge conservation and angular momentum conservation, we discuss the Hawking radiation of the Kerr–Newman black hole by the Hamilton–Jacobi method. The result shows the tunneling probability is related to the change of Bekenstein–Hawking entropy and the radiation spectrum deviates from the purely thermal one, which is full in accordance with Parikh and Wilczek's result.

The line element of the Kerr–Newman black hole [15] is given by

$$ds^2 = -\left(1 - \frac{2Mr - Q^2}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left[(r^2 + a^2) + \frac{(2Mr - Q^2)a^2\sin^2\theta}{\rho^2}\right]\sin^2\theta d\varphi^2 - \frac{2(2Mr - Q^2)a\sin^2\theta}{\rho^2}dtd\varphi, \quad (1)$$

with the electromagnetic potential

$$A_\mu = \left(\frac{Qr}{\rho^2}, 0, 0, -\frac{Qra\sin^2\theta}{\rho^2}\right), \quad (2)$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + Q^2 + a^2$. Due to the dragging effect, it isn't convenient for us to discuss the Hawking radiation of the black hole. An effective approach to describe the Hawking radiation should be in a dragging coordinate system. Besides, the event horizon $r_h = M + \sqrt{M^2 - Q^2 - a^2}$ and the outer infinite red-shift surface $r_{TLS} = M + \sqrt{M^2 - Q^2 - a^2\cos^2\theta}$ should be coincident with each other. Therefore, the dragging coordinates transformation

$$\frac{d\varphi}{dt} = \frac{(2Mr - Q^2)a}{\rho^2(r^2 + a^2) + (2Mr - Q^2)a^2\sin^2\theta} \quad (3)$$

is performed on the line element (1). And then the new line element takes on form as

$$ds^2 = -\frac{\rho^2\Delta}{(r^2 + a^2)^2 - \Delta a^2\sin^2\theta}dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2, \quad (4)$$

with the electromagnetic potential $A_\mu = (A_t, 0, 0)$, where

$$A_t = \frac{Qr(r^2 + a^2)}{(r^2 + a^2)^2 - \Delta a^2\sin^2\theta}. \quad (5)$$

The line element (4) denotes a 3-dimension hyper-surface in the 4-dimensional space-time. In the line element (4), the metrics satisfy Landau's condition of the coordinate clock synchronization, which is helpful to discuss the Hawking radiation of the black hole. The event horizon and the outer infinite red-shift surface are coincident with each other, which mean the geometrical optics limit can be relied here. Using the WKB approximation, we can get the relationship between the tunneling probability and the imaginary part of the radiation particle's action as $\Gamma \sim e^{-2\,\mathrm{Im}\, I}$. In the discussion of the Hawking radiation, the derivation of the action is crux. There are two methods to derive it. One is radial geodesic method put forward by Parikh and Wilczek. In this method, in order to get the action, one should introduce the Painlevé coordinates and explore the motion equations of radiation particles. Moreover, massless and massive particles should be differentiated for their motion equations are different. In fact, these complicated procedures can be avoided in another method, namely Hamilton–Jacobi method. The Hamilton–Jacobi method was applied extensively to the non-thermal radiation in 1990s and attracted people's attention again recently [2, 8, 21]. In this paper, we adopt the Hamilton–Jacobi method to explore the action. The action can be derived from the line element (1) and (4) respectively. For the convenience, we select the line element (4) and make a treatment to it.

Near the event horizon, the line element (4) takes on form as

$$ds^2 = -\frac{\rho^2(r_h)\Delta_{,r}(r_h)(r-r_h)}{(r_h^2+a^2)^2}dt^2 + \frac{\rho^2(r_h)}{\Delta_{,r}(r_h)(r-r_h)}dr^2 + \rho^2(r_h)d\theta^2. \tag{6}$$

In which $\rho^2(r_h) = r_h^2 + a^2\cos^2\theta$ and $\Delta_{,r}(r_h) = \frac{\partial\Delta}{\partial r}|_{r=r_h} = 2r_h - 2M$. The corresponding electromagnetic potential is $A_\mu = (\frac{Qr_h}{r_h^2+a^2}, 0, 0)$. The action of the radiation particle satisfies relativistic Hamilton–Jacobi equation as

$$g^{\mu\nu}(\partial_\mu I + qA_\mu)(\partial_\nu I + qA_\nu) + u^2 = 0, \tag{7}$$

where u, q and I are the mass, charge and action of the particle respectively, $g^{\mu\nu}$ are the inverse metric tensors obtained from the line element (6). Substituting them into (7) yields

$$-\frac{(r_h^2+a^2)^2}{\rho^2(r_h)\Delta_{,r}(r_h)(r-r_h)}(\partial_t I + qA_t)^2 + \frac{\Delta_{,r}(r_h)(r-r_h)}{\rho^2(r_h)}(\partial_r I)^2 + \frac{1}{\rho^2(r_h)}(\partial_\theta I)^2 + u^2 = 0. \tag{8}$$

It is difficult to solve the action from (8) directly. Considering the properties of the black hole space-time, we carry out the separation of variables as

$$I = -\omega t + R(r) + \Theta(\theta) + j\varphi, \tag{9}$$

where ω is the energy of the particle, $R(r)$and $\Theta(\theta)$ are the generalized momentums, and j is the angular momentum with respect to the angular φ. Substituting (9) into (7), we get

$$\begin{aligned} R_\pm(r) &= \pm\frac{r_h^2+a^2}{\Delta_{,r}(r_h)}\int\frac{dr}{r-r_h}\sqrt{(\omega - j\Omega_h - qA_h)^2 - \frac{\rho^2(r_h)\Delta_{,r}(r_h)(r-r_h)}{(r_h^2+a^2)^2}\left[\frac{[\partial_\theta\Theta(\theta)]^2}{\rho^2(r_h)} + u^2\right]} \\ &= \pm\frac{i\pi(r_h^2+a^2)(\omega - j\Omega_h - qA_h)}{\Delta_{,r}(r_h)}, \end{aligned} \tag{10}$$

where $\Omega_h = \frac{d\varphi}{dt}|_{r=r_h} = \frac{a}{r_+^2+a^2}$ and $A_h = \frac{Qr_h}{r_h^2+a^2}$ express the angular velocity and electromagnetic potential at the event horizon respectively and $+$ $(-)$ is the solution corresponding to the outgoing (incoming) particle [1, 11]. Inserting (10) into (9), we can obtain the imaginary parts of the outgoing and incoming particles' actions as

$$\operatorname{Im} I_\pm = \pm\pi \frac{(r_h^2 + a^2)(\omega - j\Omega_h - qA_h)}{\Delta_{,r}(r_h)}. \tag{11}$$

So the tunneling probability of the radiation particles is

$$\Gamma = \frac{e^{-2\operatorname{Im} I_+}}{e^{-2\operatorname{Im} I_-}} = e^{-4\pi \frac{(r_h^2+a^2)(\omega - j\Omega_h - qA_h)}{\Delta_{,r}(r_h)}}. \tag{12}$$

However, we can easily find the radiation spectrum is only leading term. The reason is that the back-reaction effect wasn't taken into account [14]. In fact, due to the black hole radiation and self-gravitational interaction, the background space-time isn't fixed and the event horizon should shrink with the particle emission. Now let's incorporate these and move on discussing the Hawking radiation of the black hole. Fixing the ADM mass, charge and angular momentum of the total space-time and allow these of the black hole to fluctuate. When a particle with energy ω, charge q and angular momentum j tunnels out, the mass, charge and angular momentum of the black hole should be replaced by $M - \omega$, $Q - q$ and $J - j$. Taking the self-gravitational interaction into account, the imaginary part of the action should be changed. Therefore the actual tunneling probability is

$$\begin{aligned}\Gamma &= \exp\left\{-4\pi \int_{(0,0,0)}^{(\omega,j,q)} \frac{(r_h'^2 + a^2)(d\omega' - \Omega_h' dj' - A_h' dq')}{\Delta'_{,r}(r_h')}\right\}\\ &= \exp\left\{2\pi \int_{(M,J,Q)}^{(M-\omega,J-j,Q-q)} \frac{(r_h'^2 + a^2)[d(M-\omega') - \frac{a}{r_h'^2+a^2} d(J-j') - \frac{(Q-q')r_h'}{r_h'^2+a^2} d(Q-q')]}{r_h' - (M-\omega')}\right\},\end{aligned} \tag{13}$$

where

$$J - j' = (M-\omega')a, \qquad r_h' = (M-\omega') + \sqrt{(M-\omega')^2 - (Q-q')^2 - a^2}. \tag{14}$$

So there is

$$\begin{aligned}\Gamma &= \exp\left\{2\pi \int_{(M,Q)}^{(M-\omega,Q-q)} \frac{r_h'^2 d(M-\omega') - r_h'(Q-q')d(Q-q')}{\sqrt{(M-\omega)^2 - (Q-q)^2 - a^2}}\right\}\\ &= \exp\left\{\pi\{[(M-\omega) + \sqrt{(M-\omega)^2 - (Q-q)^2 - a^2}]^2 - [M + \sqrt{M^2 - Q^2 - a^2}]^2\}\right\}\\ &= e^{\Delta S_{BH}},\end{aligned} \tag{15}$$

where $\Delta S_{BH} = S_{BH}(M-\omega, Q-q) - S_{BH}(M, Q)$is the change of Bekenstein–Hawking entropy before and after the charged particle emission. The result is full consistent with Ref. [28] and gives a correction to the Hawking radiation the Kerr–Newman black hole.

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